SARD'S THEOREM AND WHITNEY EMBEDDING: A GATEWAY

1. Preliminaries

Theorem 1.1. *Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is an indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Proof. Suppose M has no boundary. Then each X_α admits a basis of regular coordinate balls, and therefore $\mathcal{X}$ admits a countable, locally finite refinement $\{B_i\}$, where $\{\overline{B}_i\}$ is also locally finite. Then there exist $\varphi_i : B_i' \to \mathbb{R}^n$ fixing $\varphi_i(\overline{B}_i) = \overline{B}_{r_i}(0)$. Define smooth functions f_i

$$f_i = \begin{cases} H_i \circ \varphi_i & \text{on } B_i' \\ 0 & \text{on } M \setminus \overline{B}_i \end{cases},$$

where $H_i : \mathbb{R}^n \to \mathbb{R}$ is smooth and positive on $B_{r_i}(0)$ and zero elsewhere. Then $f(x) := \sum_i f_i(x)$ is well-defined and smooth, and $g_i(x) := f_i(x)/f(x)$ are smooth. It is straightforward to check $0 \leq g_i \leq 1$, and $\sum_i g_i \equiv 1$.

Reindexing, we set

$$\psi_\alpha = \sum_{i : B_i' \subseteq X_\alpha} g_i.$$

After a few further straightforward checks, the result follows. □

One immediate application of partitions of unity is the existence of smooth bump functions, which will play important roles in the proofs of Sard's Theorem, the Whitney Embedding theorem, and the Whitney Approximation Theorem for functions, among others. They allow us to glue together locally defined smooth functions into global ones, and serve as a continuous analogue of indicator functions.

Theorem 1.2. *Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.*

Proof. Take a partition of unity $\psi_1, \psi_2 : M \to \mathbb{R}$ subordinate to the cover $M = X_1 \cup X_1 := U \cup (M \setminus A)$. In particular, $\psi_1 : M \to \mathbb{R}$ satisfies $\psi_1|_A \equiv 1$, $\operatorname{supp}(\psi_1) \subseteq U$, and $0 \leq \psi_1 \leq 1$ on M. □

In particular, we can realize closed subets of smooth manifolds as level sets of smooth nonnegative functions $f : M \to \mathbb{R}$.

Theorem 1.3. *Let M be a smooth manifold. If K is any closed subset of M, there is a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

There are several special families of smooth maps, one such being the smooth exhaustion functions.

Definition 1.4. A *smooth exhaustion function* is a smooth function $f : M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$, called a *sublevel set*, is compact for each $c \in \mathbb{R}$.

The sublevel sets $f^{-1}((-\infty, c])$ thereby form a compact exhaustion of M. The advantage of such a function is that noncompact manifolds M can be made to everywhere "look" compact locally. For such a smooth exhaustion function, it also holds that the sublevel sets $f^{-1}(-\infty, b])$ and $f^{-1}([a, b])$ are regular domains for a, b regular values for f.

Definition 1.5. A *regular domain in* M is a properly embedded codimension-0 submanifold with boundary, e.g. $\mathbb{H}^n \subseteq \mathbb{R}^n$.

For a smooth map $F : M \to N$, it is necessary to investigate the rank of $dF_p : T_pM \to T_{F(p)}N$ at each $p \in M$, as it informs the "non-degeneracy" of our function in local neighborhoods.

Definition 1.6. For $F : M \to N$ a smooth map, a point $p \in M$ is called a *regular point* of F if $dF_p : T_pM \to T_{F(p)}N$ is surjective, and a *critical point* otherwise.

This definition coincides with our intuition from one-variable calculus, that critical points are the places where the tangent approximation to a function is constant.

Smooth constant-rank maps serve as a family of smooth maps which admit a canonical coordinate representation described in the following *Rank Theorem*. This is a non-linear analogue of a constant-rank result for linear maps.

Theorem 1.7. *Suppose* M *and* N *are smooth manifolds of dimensions* m *and* n*, respectively, and* $F : M \to N$ *is a smooth map with constant rank* r*. For each* $p \in M$*, there exist smooth charts* (U, φ) *containing* p *and* (V, ϕ) *containing* $F(p)$ *for which* F *has the coordinate representation*

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

Proof. Reduce to the case $F : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^n$. □

We review another result which identifies smooth constant-rank maps as an important source of submanifolds.

Theorem 1.8. *Let* M *and* N *be smooth manifolds, and let* $F : M \to N$ *be a smooth map with constant rank* r*. Then each level set of* F *is a properly embedded submanifold of codimension* r *in* M*.*

Proof. Write $m = \dim(M)$, $n = \dim(N)$, and $k = m - r$. For each $p \in F^{-1}(c)$ and chart (U, φ) containing p, $F^{-1}(c) \cap U$ is the slice

$$\{(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \in U : x^1 = \cdots = x^r = 0\}.$$

Hence $F^{-1}(c)$ satisfies the local k-slice condition, and the result follows by $F^{-1}(c)$ closed. □

We recall the notion of a negligible set from measure theory, and extend this notion to abstract manifolds in the very natural way via charts.

Definition 1.9. For M a smooth n-manifold with or without boundary, $A \subseteq M$ has *measure zero* in M if for every smooth chart (U, φ) for M, the subset $\varphi(A \cap U) \subset \mathbb{R}^n$ has n-dimensional measure zero.

The following is a techncal lemma which will assist in the proof of Sard's Theorem to come. It lifts "thinness" of compact sets on $(n-1)$-dimensional slices to n-dimensional space.

Lemma 1.10. *Let $A \subseteq \mathbb{R}^n$ be a compact subset whose intersection with $\{c\} \times \mathbb{R}^{n-1}$ has $(n-1)$-dimensional measure zero for every $c \in \mathbb{R}$. Then A has n-dimensional measure zero.*

2. Sard's Theorem: Statement, Proof, Applications

Proven in full generality by Arthur Sard in 1942, Sard's Theorem is a remarkable result underlying celebrated theorems such as the Whitney Embedding Theorem, Parametric Transversality Theorem, and Transversality Homotopy Theorem which study embeddings and intersections of submanifolds.

Theorem 2.1. *Suppose M and N are smooth manifolds with or without boundary and $F : M \to M$ is a smooth map. Then the set of critical values of F has measure zero in N.*

Proof. (Sketch) Set $m = \dim(M)$, $n = \dim(N)$. We proceed by induction on $m \in \mathbb{Z}_{\geq 0}$. Denote C the collection of critical points of F.

When $m = 0$, the collection of critical points is either empty or countable. In either case, $F(C)$ has measure zero in N.

Let $m > 0$. Covering M by countably many charts, we may reduce to the case $F : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$. Now, define the following sets:

$$C_k := \{x \in C : \text{ all } k\text{th order partical derivatives of } F \text{ vanish at } x\},$$

where $C \supseteq C_1 \supseteq C_2 \supseteq \dots$.

Each C_k is closed by continuity. Next, we show the following:

(1) $F(C \setminus C_1) \subseteq N$ has measure zero,
(2) $F(C_k \setminus C_{k+1})$ has measure zero for all k, and
(3) $F(C_k)$ has measure zero for k sufficiently large.

Toward (1), pick any $a \in C \setminus C_1$, and without loss of generality let $\partial F^1/\partial x^1$ be nonzero at a. Then we may change coordinates in some neighborhood V_a from $(x^1, \dots, x^m)$ to $(F^1, x^2, \dots, x^m)$. In this new coordinate representation, dF takes the form

$$\begin{pmatrix} 1 & 0 \\ * & \partial F^i/\partial x^j \end{pmatrix},$$

where $(\partial F^i/\partial x^j)_{ij}$ has rank $< n-1$ on $C \cap \overline{V}_a$. The previous technical lemma together with an application of the induction hyptohesis establishes that $F(C \cap \overline{V}_a)$ and therefore $F(U \cap C)$ has measure zero.

Now to prove (2), fix $a \in C_k$, and let $y : U \to \mathbb{R}$ be a kth partial derivative of F which admits a non-zero partial derivative at a. Then we again obtain a neighborhood $V_a \ni a$ consisting of regular points. Then $C_k \cap V_a \subseteq y^{-1}(0) \cap V_a$, hence $F(C_k \cap V_a)$ constitute critical values of $F|_{y^0 \cap V_a}$, which by the induction hypothesis have measure zero. Again, countably many $F(C_k \cap V_a)$ cover $F(C_k \setminus C_{k+1})$.

Lastly, we establish (3). Roughly, we can bound the absolute value of all $(k+1)$st partial derivatives of F in any closed cube E. Subdividing a E into sufficiently small subcubes E_i bounds–by a version of Taylor's theorem–$F(C_k \cap E_i)$ within open balls, the sum of whose volumes may be made negligible.

Since $F(C)$ is comprised of the above sets (of which there are countably many), the result follows.

□

We proceed to a few interesting applications and further comments. Two corollaries include the following, which incorporate dimension.

Corollary 2.2. *Suppose M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map. If dim$(M) <$ dim(N), then $F(M) \subseteq N$ has measure zero.*

Proof. In this case, every point in M is critical for F. □

Corollary 2.3. *Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold with or without boundary. If dim$(S) <$ dim(M), then $S \subseteq$ has measure zero.*

Proof. The above follows from the previous corollary applied to the inclusion $i : S \to M$. □

Sard's theorem is also used to establish the existence of *Morse functions*, or functions $F : M \to \mathbb{R}$ all of whose critical points are *non-degenerate*. Such functions carry rich topological information.

3. The Whitney Embedding Theorem: Statement, Proof, Applications

Established in the 1930s by Hassler Whitney, the Whitney Embedding Theorem is a fundamental result in differential topology boasting wide application. It confirms the natural impulse to visualize abstract manifolds as embedded in a Euclidean space.

Theorem 3.1. *Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Proof. We will first show that M admits a smooth embedding into *some* Euclidean space. Suppose M is compact. Then M admits a finite cover $\{B_1, \ldots, B_m\}$ from which we may generate coordinate charts (B_i', φ_i), $B_i' \supseteq \overline{B}_i$ and $\varphi_i : B_i' \to \mathbb{R}^n$. For smooth bump functions $\rho_i : M \to \mathbb{R}$, $\rho_i|_{\overline{B_i}} \equiv 1$, consider the smooth map $F : M \to \mathbb{R}^{nm+m}$ given by

$$F(p) = (\rho_1\varphi_1(p), \ldots, \rho_m\varphi_m(p), \rho_1(p), \ldots, \rho_m(p)) .$$

By M compact, it suffices to show F is an injective, smooth immersion. Injectivity follows by $\{B_1, \ldots, B_m\}$ a cover for M and the φ_i homeomorphisms. The immersion property follows by $\{B_1, \ldots, B_m\}$ a cover for M and $d(\rho_i\varphi_i)_p = d(\varphi_i)_p$ in some neighborhood of any $p \in B_i$, which is injective by φ_i a diffeomorphism on $\overline{B}_i$.

Suppose M is non-compact. Take a smooth exhaustion function $f : M \to \mathbb{R}$ of M, and, for regular values $i < a_i < b_i < i+1$ guaranteed for all $i \in \mathbb{Z}$ by Sard's theorem, define subsets $D_0 = f^{-1}((-\infty, 1])$, $D_i = f^{-1}([i, i+1])$, $E_1 = f^{-1}((-\infty, a_1])$, and $E_i = f^{-1}([a_{i-1}, b_{i+1}])$. Notice that $D_i \subseteq \mathit{Int}(E_i)$. The argument above guarantees an embedding $\varphi_i : E_i \to \mathbb{R}^{nm+m}$. Then let $\rho_i : M \to \mathbb{R}$ be a smooth bump function satisfying $\rho_i|_{D_i} \equiv 1$, $\operatorname{supp}(\rho_i) \subseteq \operatorname{Int}(E_i)$. Define

$$F(p) = \left(\sum_{i \text{ even}} \rho_i(p)\varphi_i(p), \sum_{i \text{ odd}} \rho_i(p)\varphi_i(p), f(p) \right).$$

This function is easily seen to be smooth, proper, injective, and finally an immersion. The result follows.

□

We proceed to a discussion of the applications of the Whitney Embedding Theorem. A first application is to the Whitney Approximation Theorem, which asserts that continuous maps $f : M \to N$ are homotopic to smooth maps. That is, they may be deformed continuously into smooth maps. The Whitney Embedding Theorem will also be used to establish the existence of a *complete Riemannian metric* on every connected smooth manifold. Such a metric enriches the underlying smooth manifold with a geometry, making familiar notions like length, angle, and distance meaningful. The theory of Riemannian geometry is especially relevant for adjacent disciplines like physics; for example, a certain generalization of Riemannian metrics plays a central role in general relativity theory. By situating abstract manifolds into a tangible Euclidean setting, Whitney's theorem broadly facilitates a dialogue between abstract and physical theory.